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Automorphism groups of linear spaces and their parabolic subgroups ☆

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ABSTRACT

Suppose that an almost simple group G acts line transitively on a finite linear space $\mathcal{S}$. Let G_x be a point stabilizer in G and suppose that G has socle T, a simple group of Lie type. In this paper we show that if $T \cap G_x$ is a parabolic subgroup of T, then G is flag transitive on $\mathcal{S}$.

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In this paper we prove the following theorem:

Main Theorem. *Suppose that $\mathcal{S}$ is a finite linear space and G is an almost simple group acting on $\mathcal{S}$ line transitively. Let G_x be the stabilizer in G of a point x of $\mathcal{S}$ and suppose the socle T of G is a simple group of Lie type. If the intersection of G_x and T is a parabolic subgroup of T, then G acts on $\mathcal{S}$ flag transitively.*

Thus by the classification of flag transitive finite linear spaces given in [3], we have the following conclusion.

Corollary. *If a pair $(\mathcal{S}, G)$ satisfies all conditions of the above theorem, then one of the following three cases occurs:*

(1) *$\mathcal{S} = PG(n,q)$ and the socle of G is isomorphic to $PSL(n+1,q)$ for some $n \geqslant 2$ and a prime power q.*
(2) *$\mathcal{S}$ is the Hermitian unital of order q and the socle of G is isomorphic to $PSU(3,q)$, where q is a prime power.*
(3) *$\mathcal{S}$ is the Ree unital of order q and the socle of G is isomorphic to ${}^2G_2(q)$, where $q = 3^{2m+1}$ for some m.*

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Obviously, this paper is part of the project to classify all line transitive finite linear spaces, that is, to classify all pairs $(\mathcal{S}, G)$, where $\mathcal{S}$ is a finite linear space and G is a group which acts line transitively on $\mathcal{S}$. According to [8], such a group G takes one of three forms:

(1) It contains a normal subgroup acting intransitively on the set of points.
(2) It contains an elementary abelian subgroup acting regularly on the set of points.
(3) It is almost simple, that is, $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$ for a non-abelian simple subgroup T.

Up to now, most of the work done on this classification problem has dealt with groups belonging to the third (3) category, and for which the socle T of G is a group of Lie type. Moreover, in some of those cases the rank of T is assumed to be high, while in other cases it is assumed to be very low. In this paper, we consider groups belonging to this third category, with socle T of Lie type, but we give a result which does not depend on the rank of the simple group T.

This paper is divided into two sections. In the first one, we recall some preliminary concepts and give some lemmas that will be used in the proof of the main theorem. The proof of the main theorem is given in Section 2. The notation on group theory used in this paper is standard. Most of the results from the theory of groups of Lie type used in the proof can be found in Carter's books [4] and [5]. Also, the notation for groups of Lie type such as $B, U, H, N.P_J, U_J, L_J, \ldots,$ for root systems, and for Weyl groups such as $\Phi, \Pi, \Phi_J, \Phi^+, W, W_J, \ldots,$ is the same as in [4].

1. Preliminaries

A *linear space* is a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set whose elements are called *points*, and $\mathcal{L}$ is a collection of some distinguished subsets of $\mathcal{P}$ called *lines*, such that any two points lie on exactly one line. The number of points on a line L is called the *length* of L. A linear space S is called *regular* if all lines in $\mathcal{L}$ have the same length.

A permutation g of $\mathcal{P}$ is called an *automorphism* of $\mathcal{S}$, if g sends every line (as a set of points) to a line. All automorphisms of a linear space $\mathcal{S}$ form a group $\mathrm{Aut}(\mathcal{S})$, called the *automorphism group* of $\mathcal{S}$. A subgroup G of $\mathrm{Aut}(\mathcal{S})$ is said to be *point transitive* if G is transitive on $\mathcal{P}$, and G is *line transitive* if it is transitive on $\mathcal{L}$. *Point primitivity* and *line primitivity* are similarly defined. For a line L and a point $x \in L$, the pair (x, L) is called a *flag*. If G is transitive on the set of all flags, then G is said to be *flag transitive*. Note that if $G \leqslant \mathrm{Aut}(\mathcal{S})$ is line transitive for some G, then all lines have the same length and $\mathcal{S}$ is regular. In this case, if all lines have the same length k and if there are v points and b lines in $\mathcal{S}$, then for any given point x, there are exactly $r = \frac{bk}{v}$ lines which contain x. Also we have

$$bk(k-1) = v(v-1).$$

Lemma 1. *If a regular linear space $\mathcal{S}$ has v points and b lines, then $v \leqslant b$.*

Lemma 2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space and $G \leqslant \mathrm{Aut}(\mathcal{S})$.*

(i) *(Block [1].) If G is line transitive, then G is point transitive.*
(ii) *(Higman and Mclaughlin [13].) If G is flag transitive, then G is point primitive.*

In the following we always assume that G is a line transitive subgroup of $\mathrm{Aut}(\mathcal{S})$. For any $L \in \mathcal{L}$, we use G_L to denote the set-wise stabilizer of L in G. Thus $b = |G : G_L|$. Let $G_{(L)}$ be the point-wise stabilizer of L in G, then G_L induces a group $G^L = G_L/G_{(L)}$ which acts on points of L. For any point $x \in \mathcal{P}$, let G_x be the stabilizer of x in G. Then $v = |G : G_x|$. Since G is line transitive, and so G is point transitive, the stabilizers of different lines are conjugate, and the stabilizers of different points are conjugate.

Let (u, v) be the greatest common divisor of two integers u and v. Define

$$b_1 = (b, v), \qquad b_2 = (b, v-1),$$

$$k_1 = (k, v), \qquad k_2 = (k, v-1).$$

Then the following facts are obvious:

Lemma 3.

$$b = b_1 b_2, \qquad k = k_1 k_2, \qquad v = b_1 k_1, \qquad r = b_2 k_2,$$

and

$$v - 1 = k_2 b_2 (k-1), \quad b_2 \geqslant k_1.$$

In [9] a prime p is said to be a *significant prime* for a finite linear space $\mathcal{S}$ if $p|(b, v-1)$. Thus the significant primes are just the prime divisors of b_2.

Now let $\mathcal{F}$ denote the set of all flags of $\mathcal{S}$. If $x \in L$, then $x^g \in L^g$ for any $g \in G$ and this yields an action of G on the set $\mathcal{F}$. Suppose $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_l$ are the orbits of G on $\mathcal{F}$. For a line L, let $\mathcal{P}(L)$ be the set of all points in L. For $1 \leqslant i \leqslant l$, let

$$\mathcal{P}^i(L) = \{x \mid (x, L) \in \mathcal{F}_i\}.$$

Then $\mathcal{P}^i(L)$ is clearly an orbit of G^L on $\mathcal{P}(L)$. Since for any $x \in L$, (x, L) is in some $\mathcal{F}_j$, we know that

$$\mathcal{P}(L) = \mathcal{P}^1(L) \cup \cdots \cup \mathcal{P}^l(L)$$

is a disjoint union and $\{\mathcal{P}^1(L), \ldots, \mathcal{P}^l(L)\}$ contains all orbits of G^L on $\mathcal{P}(L)$. Moreover, it is easily seen that $(\mathcal{P}^i(L))^g = \mathcal{P}^i(L^g)$ for any $g \in G$. Thus $|\mathcal{F}_i| = b|\mathcal{P}^i(L)|$. Since v divides $|\mathcal{F}_i|$, we know that k_1 divides $|\mathcal{P}^i(L)|$. We thus obtain the following lemma:

Lemma 4. *Suppose G has l orbits on the flag set $\mathcal{F}$. Then G_L has l orbits on the set $\mathcal{P}(L)$ and the length of every orbit is a multiple of k_1. In particular, G is flag-transitive if and only if G_L is transitive on the set $\mathcal{P}(L)$.*

Let X be a set. We define $X^{(2)} = \{(x, y) \mid x \neq y \in X\}$. If H is a permutation group on X, then H acts on the set $X^{(2)}$. Let $\psi_1, \ldots, \psi_t$ be the orbits of G^L on $\mathcal{P}(L)^{(2)}$ and $\Psi_1, \ldots, \Psi_s$ be the orbits of G on $\mathcal{P}^{(2)}$. We define a map ρ from the set $\{\psi_1, \ldots, \psi_t\}$ to $\{\Psi_1, \ldots, \Psi_s\}$ by sending ψ_i to Ψ_j if $\psi_i \subseteq \Psi_j$. It is easily seen that this map ρ is a bijection from $\{\psi_1, \ldots, \psi_t\}$ to $\{\Psi_1, \ldots, \Psi_s\}$, so $t = s$. We may assume that $\rho: \psi_i \to \Psi_i$ for $i = 1, \ldots, t$. If $(a, b) \in \psi_i \subseteq \Psi_i$, then $|\Psi_i| = |G : G_{a,b}| = |G : G_L||G_L : G_{a,b}| = b|\psi_i|$. Thus b divides $|G : G_{a,b}| = v|G_a : G_{a,b}|$, from which we deduce that b_2 divides $|G_a : G_{a,b}|$. Hence if G is line-transitive, then b_2 divides the length of every suborbit of G.

Now suppose that

$$\mathcal{P}(L) = \mathcal{P}^1(L) \cup \cdots \cup \mathcal{P}^l(L)$$

is the decomposition of $\mathcal{P}(L)$ into orbits of G^L and suppose that $l \geqslant 2$. Let $(a, b) \in \psi_j$ (so $(a, b) \in \Psi_j$), $a \in \mathcal{P}^i(L)$, and $b \in \mathcal{P}^m(L)$ for $1 \leqslant i, m \leqslant l$. If Ψ_j is self-paired, then there is an element $g \in G$ such that $(a, b)^g = (b, a)$. Thus $a^g = b$ and $b^g = a$. This means that $L^g = L$, $i = m$ and a, b are in the same orbit $\mathcal{P}^i(L)$. When we regard the group $G_L^{\mathcal{P}^i(L)}$ as a permutation group on $\mathcal{P}^i(L)$, b is in a self-paired orbit of the point stabilizer $(G_L^{\mathcal{P}^i(L)})_a$ of a. Thus we get:

Lemma 5. *Let $\psi_1, \ldots, \psi_t$ and $\Psi_1, \ldots, \Psi_s$ be as above. Then*:

(1) *The map ρ which maps ψ_i to Ψ_j if $\psi_i \subseteq \Psi_j$ is bijective.*
(2) *Every subdegree of G on $\mathcal{P}$ is a multiple of b_2.*
(3) *If Ψ_j is a self-paired orbit and $(a, b) \in \Psi_j$, where a and b are points of L, then a and b are in the same orbit of G_L on $\mathcal{P}(L)$; and if they are in $\mathcal{P}^i(L)$, then b is in a self-paired orbit of $(G_L^{\mathcal{P}^i(L)})_a$.*

Corollary 6. *If $G \leqslant \mathrm{Aut}(\mathcal{S})$ is line transitive and all suborbits of G on $\mathcal{P}$ are self-paired, then G is flag transitive.*

Proof. Let a and b be two points on a line L. Then the ordered pair (a,b) is in a self-paired orbit on $\mathcal{P}^{(2)}$. Hence a and b are in the same orbit of G^L on $\mathcal{P}(L)$. Thus G^L is transitive on $\mathcal{P}(L)$, and so G is flag transitive on $\mathcal{S}$. $\square$

Corollary 7. *Suppose that a group G acts line transitively on a linear space $\mathcal{S}=(\mathcal{P},\mathcal{L})$, and H_1 and H_2 are two subgroups of G such that $G_x \leqslant H_1 < H_2 \leqslant G$, where x is a point of $\mathcal{P}$. Then $|H_2 : H_1| \equiv 1 \pmod{b_2}$.*

Proof. Let Δ be the orbit of H_2 on $\mathcal{P}$ containing x. If $x \neq y \in \Delta$, then since $G_x \leqslant H_2$, the orbit y^{G_x} is totally contained in Δ. This means that the set $\Delta - \{x\}$ is a disjoint union of some orbits of G_x. Hence we have $|H_2 : G_x| \equiv 1 \pmod{b_2}$ by Lemma 5. Replacing H_2 by H_1 we know that $|H_1 : G_x| \equiv 1 \pmod{b_2}$. Since $|H_2 : G_x| = |H_2 : H_1| \times |H_1 : G_x|$, the conclusion follows. $\square$

The next two lemmas are about some properties of Weyl groups, which are needed in the proof of the main theorem. Let G be a group, H a subgroup of G and g an element of G. We say that the double coset HgH is *self-paired* if $HgH = Hg^{-1}H$. It is obvious that if G is represented as a permutation group on the cosets G/H of H in G, then every orbit of H on G/H corresponds to a double coset, and if an orbit Δ corresponds to HgH, then Δ is self-paired if and only if $HgH = Hg^{-1}H$.

Lemma 8. *Let W be the Weyl group of type A_n defined on the set $I=\{1,2,\ldots,n\}$. Then for any subset $J = I-\{j\}$, every (W_J, W_J)-double coset of W is self-paired.*

Proof. It is well known that $W \cong \mathrm{Sym}(n+1)$. Thus W can be viewed as the symmetric group defined on the set $I_1=\{1,2,\ldots,n+1\}$. Take $J_1=\{1,2,\ldots,j\} \subseteq I_1$. Then W_J is the stabilizer of J_1 in $\mathrm{Sym}(n+1)$. Let $g \in W\backslash W_J$, then $(W_J)^g$ is the stabilizer of $J_1{}^g = \{1^g, 2^g, \ldots, j^g\}$. Let $X = J_1 \cap J_1^g$, $Y_1 = J_1 - X$, and $Y_2 = J_1^g - X$. Since g is not in W_J, $J_1 \neq J_1^g$ and so $|X| < |J_1|$, $|Y_1| = |Y_2|$, and $Y_1 \cap Y_2 = \emptyset$. Now let $Y_1 = \{k_1, \ldots, k_t\}$ and $Y_2 = \{l_1, \ldots, l_t\}$. Take $h = (k_1, l_1)(k_2, l_2)\ldots(k_t, l_t)$, then h is an element of W of order 2. Now we take an element $i \in J_1$. If $i^g \in X$, then $i^{gh} = i^g$ is in X. If $i^g \in Y_2$, then i^{gh} is in $Y_1 \subseteq J_1$. Thus $J_1{}^{gh} = J_1$ and so $gh \in W_J$. This means that $W_J g W_J = W_J h W_J$. Since h is an involution, so $W_J g W_J = W_J h W_J$ is self-paired. $\square$

Take an n-dimensional Euclidean space $\mathcal{V}$ and let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis. Then the set

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\}$$

constitutes a root system of type D_n. Let $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\ldots$, $\alpha_{n-1} = e_{n-1} - e_n$, and $\alpha_n = e_{n-1} + e_n$. Then

$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

is a fundamental system of Φ.

Lemma 9. *Let W be the Weyl group of the root system Φ defined as above. Then for the subset $J = I - \{n\}$, every (W_J, W_J)-double coset of W is self-paired.*

Proof. The Weyl group W of Φ is generated by all reflections of the form r_α with $\alpha \in \Pi$. So W acts on the set $\{e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n\}$ imprimitively with blocks of imprimitivity $\{e_1, -e_1\}, \{e_2, -e_2\}, \ldots,$ and $\{e_n, -e_n\}$. Indeed W is the semi-direct product of a normal elementary abelian subgroup A of order 2^{n-1} and the symmetric group $\mathrm{Sym}(n)$. If $J = I - \{n\}$, then $W_J \cong \mathrm{Sym}(n)$. Let W/W_J be the set of cosets of W_J in W, then W acts on W/W_J and the normal subgroup A is regular on it. Since A is an elementary abelian 2-group of order 2^{n-1}, every (W_J, W_J) double coset is self-paired. $\square$

2. The proof of the Main Theorem

Suppose the main theorem is false and let $(\mathcal{S}, G)$ be a counterexample to the theorem. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite linear space and $G \leqslant \mathrm{Aut}(\mathcal{S})$ line transitively acts on $\mathcal{S}$, and they satisfy the following conditions:

(A) G is an almost simple group with socle T a simple group of Lie type.
(B) For a point $x \in \mathcal{P}$, the stabilizer G_x intersects T in a parabolic subgroup of T.
(C) G is not flag transitive on $\mathcal{S}$.

By the condition (B), T is either a Chevalley group or a twisted group. We assume that T is defined over a field $GF(q)$ with $q = p^a$. When G is of type ${}^2B_2$ or ${}^2F_4$, $p = 2$. When G is of type ${}^2G_2$, $p = 3$. In all other cases, p is an arbitrary prime. Finally let the rank of T be n.

We suppose the existence of the above counterexample and shall obtain a contradiction in 11 steps:

Step 1. *We may assume that the rank of T is greater than* 1.

Proof. If T is of rank 1, then T acts doubly transitively on the set $\mathcal{P}$ by hypothesis (B), and so G acts on $\mathcal{P}$ doubly transitively. Thus G is flag transitive on $\mathcal{S}$ by Corollary 6, contradicting (C). □

Step 2. *We may assume that G is simple.*

Proof. In order to prove this assertion, we need the notion of exceptionality. Let L be a normal subgroup of a group H which acts on a set $\mathcal{P}$. Then $(H, L, \mathcal{P})$ is called exceptional if the only common orbital of H and L in their actions on $\mathcal{P}$ is the diagonal (see [12]). Now we show that the socle T of G is line transitive on $\mathcal{S}$. Suppose that T is not line transitive on $\mathcal{S}$, then since G/T is solvable, there is a subgroup series

$$T = A_1 \lhd A_2 \lhd \cdots \lhd A_s = G,$$

such that every quotient A_{i+1}/A_i is a cyclic group of prime order. So there are two subgroups H and L in G such that $T \leqslant L \lhd H$, $|H : L|$ is a prime, and H is line transitive but L is not. Then by [10], either $\mathcal{S}$ is a projective plane or $(H, L, \mathcal{P})$ is an exceptional triple. If $\mathcal{S}$ is a projective plane, then $G \geqslant PSL(3, q)$ acts flag transitive on $\mathcal{S}$ by the result of [11] and hence $(\mathcal{S}, G)$ is not a counterexample, a contradiction.

Suppose that $(H, L, \mathcal{P})$ is an exceptional triple. If H acts primitively on $\mathcal{P}$, then by Theorem 1.5 in [12], as the rank of G is greater than 1, we know that $T \cap G_x$ is a subfield group, contradicting the hypothesis (B). If H acts on $\mathcal{P}$ imprimitively, then let $\overline{\mathcal{P}}$ be a set of blocks of imprimitivity of H on $\mathcal{P}$ such that H acts on $\overline{\mathcal{P}}$ primitively. By Lemma 3.5 in [12], $(H, L, \overline{\mathcal{P}})$ is also exceptional. Applying the above Theorem in [12] to $(H, L, \overline{\mathcal{P}})$, we know that $T \cap G_{\bar{x}}$ is also a subfield group where $\bar{x}$ is the block of imprimitivity containing x. Since $G_x \leqslant G_{\bar{x}}$, this is impossible. Hence we conclude that the socle T also acts on $\mathcal{S}$ line transitively.

But if T is flag transitive, then G is also flag transitive. Thus if $(\mathcal{S}, G)$ is a counterexample, then $(\mathcal{S}, T)$ is a counterexample as well. For this reason, we may assume that in our counterexample $(\mathcal{S}, G)$, the group G is simple. □

Step 3. *If G is a Chevalley group, then G is of type A_n for some n, or of type D_n for some odd n, or of type E_6. Furthermore, if G is of type D_n, then $|J \cap \{n-1, n\}| = 1$.*

Proof. By Step 2, G is a simple group of Lie type and the point stabilizers are parabolic subgroups of G by (B). Now we suppose that G is a Chevalley group. Thus by [9], all line stabilizers are also parabolic subgroups as the rank of G is greater than 1. We fix a Borel subgroup B and choose a point

x and a line L such that both G_x and G_L contain B. Let the root system Φ associated with G be defined on the set $I = \{1, 2, \ldots, n\}$, we may assume that $G_x = P_J$ and $G_L = P_K$ for some $J, K \subseteq I$.

Consider the Weyl group W of Φ and let w_0 be the longest element of W. We first show that if w_0 sends the set Φ_{J_0} to itself for some J_0, then G_x is not equal to the parabolic subgroup P_{J_0}. In fact, the parabolic subgroup P_{J_0} has a Levi decomposition $P_{J_0} = U_{J_0} : L_{J_0}$, that is, P_{J_0} is a semi-direct product of U_{J_0} by L_{J_0}, where $U_{J_0} \leqslant U$ and

$$L_{J_0} = \langle H, X_\alpha \mid \alpha \in \Phi_{J_0} \rangle.$$

Let n_0 be a preimage of w_0 under the natural homomorphism $N \to W$. Since w_0 sends every positive root to a negative one, ${U_{J_0}}^{n_0} \leqslant V$, where V is the subgroup of G generated by all root subgroups X_α with α negative. Also, as ${\Phi_{J_0}}^{w_0} = \Phi_{J_0}$, we know that ${L_{J_0}}^{n_0} = L_{J_0}$. Thus if $G_x = P_{J_0}$ and $x^{n_0} = y$, then $G_{x,y} = G_x \cap G_y = L_{J_0}$, and so $|G_x : G_{x,y}|$ is a power of p. By Lemma 5 this means that p is a significant prime, which is impossible by the result of Gill in [9].

Now suppose that the type of Φ is not in the set $\{A_n, E_6, D_{2m+1}\}$. Then it is easily seen from the Dynkin diagram of Φ that the longest element w_0 of W sends every positive root α to its negative $-\alpha$ and so w_0 sends Φ_J to itself for every $J \subseteq I$. If G is of type D_n for some odd n and $|J \cap \{n-1, n\}| = 0$ or 2, then w_0 sends Φ_J to itself also. By the fact we just proved, we know the above assertion on the type of G is true. $\square$

Step 4. *If $(\mathcal{S}, G)$ is a counterexample to the main theorem, then G cannot be a twisted group.*

Every twisted group G has a (B, N)-pair and the Weyl group W acts on a system Φ of finitely many non-zero vectors in an ordered Euclidean space. Thus we also can speak of positive and negative vectors of Φ. The set Φ has a subset Π such that every element $\alpha \in \Phi$ can be expressed as a linear combination of elements of Π with coefficients all non-negative or all non-positive. Let $\Pi = \{\alpha_i,\ i \in I\}$ where $I = \{1, 2, \ldots, n\}$. Every parabolic subgroup corresponds to a subset J of I and the Levi decomposition is also valid for twisted groups. Since in the case of twisted groups, the longest element $w_0 \in W$ sends every positive vector of Φ to its negative, we can prove this assertion by the same method as in Step 3.

Thus in the following we only need to consider the cases where G is of type A_n, of type D_n with n odd or of type E_6. Recall that $G_x = P_J$ and $G_L = P_K$ for some $J, K \subseteq I$.

Step 5.

(1) *Let α_j and α_i be two roots in the fundamental system Π such that j is in J but i is not. If α_j is orthogonal to α_i, then $j \in K$.*
(2) *There is at least one element $j_0 \in J$ such that α_{j_0} is not orthogonal to α_i whenever $i \in I - J$.*

Proof. (1) Let r_i be the reflection determined by α_i and $n_i \mapsto r_i$ under the natural homomorphism $N \to W$. Since i is not in J, $x^{n_i} = y$ is a point in $\mathcal{P}$ different from x. Thus $P_J \cap P_J^{n_i} = G_x \cap G_y$ is contained in the stabilizer G_M for some line M. Because of line transitivity, there is an element $g \in G$ such that $(P_J \cap P_J^{n_i})^g \leqslant P_K$. Since U_{α_i} is a p-subgroup of $P_J \cap P_J^{n_i}$ and U is a Sylow p-subgroup of P_K, we may assume that $(U_{\alpha_i})^g \leqslant U$. On the other hand, we can write $g = b_1 n b_2$ for some $b_1, b_2 \in B$ and $n \in N$. So we have $(U_{\alpha_i})^{b_1 n b_2} \leqslant U$. This implies that $(U_{\alpha_i})^n \leqslant U$. Thus the image w of n under the homomorphism $N \to W$ is in $\langle r_i \rangle$, and we know that either $g \in B$ or $g \in B n_i B$. If $g \in B$, then clearly $P_{\{j\}} \leqslant P_K$. Now suppose that $g \in B n_i B$. Then g can be written as $g = u n_i b$ for some $b \in B$ and u in the root subgroup X_{α_i}. Since α_i is orthogonal to α_j, we know that u commutes with elements in X_{α_j} and elements in $X_{-\alpha_j}$ by the Chevalley's commutator formula. Also, since α_i is orthogonal to α_j, $X_{\pm\alpha_j}^{n_i} = X_{\pm\alpha_j}$. Hence we have $P_{\{j\}} \leqslant P_K$, which implies $j \in K$.

(2) We observe that if $J \subseteq K$, then $P_J \leqslant P_K$, and so $\mathcal{S}$ is a projective plane. This is impossible. Hence the set J is not a subset of K, and the conclusion of (1) implies the existence of the element j_0 in the claim (2). $\square$

In Steps 6–9, we deal with the cases where G is of type A_n or of type D_n.

Step 6. *Suppose that G is of type A_n or D_n, then it is impossible that $|I - J| = 1$. In other words, G_x is not a maximal parabolic subgroup.*

Proof. Suppose that $|I - J| = 1$. Then in the A_n case, $J = I - \{i\}$ for some i. If G is of type D_n, then since the symmetry of the Dynkin diagram which interchanges nodes $n - 1$ and n determines a graph automorphism of G and since $|J \cap \{n-1, n\}| = 1$, we may assume that $J = I - \{n\}$. Thus in both cases, every (W_J, W_J)-double coset in W is self-paired by Lemmas 8 and 9. Now we show that if this happens, then every (P_J, P_J)-double coset in G is also self-paired: Every (P_J, P_J)-double coset can be written in the form $P_J n P_J$, for some $n \in N$. Let n maps to w under the natural homomorphism $N \to W$. Since $W_J w W_J$ is self-paired, we have $w^{-1} \in W_J w W_J$. Let N_J be the preimage of W_J under the homomorphism $N \to W$. Since under this homomorphism, n^{-1} maps to w^{-1}, we know that $n^{-1} \in N_J n N_J \subseteq P_J n P_J$. Hence $P_J n P_J = P_J n^{-1} P_J$ and $P_J n P_J$ is self-paired. Since $G_x = P_J$, we know that every $G_x g G_x$ is self-paired and G is flag transitive by Corollary 6, a contradiction. □

Step 7. *We claim that: If G is of type A_n, then J has the form $I - \{s, s+2\}$. If G is of type D_n, then $J = I - \{n-3, n\}$.*

Proof. By Step 6 we know that $|I - J| > 1$. In a Dynkin diagram of type A_n, every node has valence at most 2. If $|I - J| \geqslant 3$, then by Step 5, we have $J \subseteq K$, which is impossible. So $|I - J| = 2$. Also, by Step 5, there is an α_j with $j \in J$ such that for any $i \in I - J$, α_j is not orthogonal to α_i. Thus we have $I - J = \{j-1, j+1\}$. Set $s = j - 1$, we get $J = I - \{s, s+2\}$.

We have shown that in the case of D_n, $|J \cap \{n-1, n\}| = 1$. Thus we may assume that $n - 1 \in J$ and $n \notin J$. Since every α_i with $i \neq n-2$ and $i \neq n$ is orthogonal to α_n, we have $n-2 \in J$, and $J - \{n-2\}$ is contained in K. Since $|I - J| \geqslant 2$ the element $n - 3$ must contained in $I - J$. In this case if $|I - J| \geqslant 3$, then we will have $J \subseteq K$, which is impossible. Thus we have that $J = I - \{n-3, n\}$. □

Step 8. *There is no counterexample $(\mathcal{S}, G)$ to the main theorem where G is of type D_n with n odd.*

Proof. Suppose that G is of type D_n with n odd. Since n is odd, $n \geqslant 5$. In this case $G_x = P_J$ where $J = I - \{n-3, n\}$. We consider the Weyl group of G. We take an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of an Euclidean space $\mathcal{V}$, and consider the root system Φ of type D_n and the fundamental root system Π defined in Lemma 9. The Weyl group W is generated by all reflections r_α with $\alpha \in \Pi$. Now $J = I - \{n-3, n\}$ and thus

$$\Pi_J = \{e_1 - e_2, \dots, e_{n-4} - e_{n-3}\} \cup \{e_{n-2} - e_{n-1}, e_{n-1} - e_n\},$$

and $\Phi_J = \Phi_{J,1} \cup \Phi_{J,2}$, where

$$\Phi_{J,1} = \{\pm(e_i - e_j) \mid 1 \leqslant i < j \leqslant n-3\}$$

and

$$\Phi_{J,2} = \{\pm(e_i - e_j) \mid n-2 \leqslant i < j \leqslant n\}.$$

Since n is odd, $n - 3$ is even. Now we define an orthogonal transformation w of $\mathcal{V}$ by $w(e_i) = -e_i$ when $1 \leqslant i \leqslant n-3$ and $w(e_i) = e_i$ when $n-2 \leqslant i \leqslant n$. Then $w \in W$ but $w \notin W_J$. We have $\alpha^w = -\alpha$ if $\alpha \in \Phi_{J,1}$ and $\alpha^w = \alpha$ if $\alpha \in \Phi_{J,2}$. Hence $\Phi_J^w = \Phi_J$, and $L_J^n = L_J$ for a preimage n of w under the homomorphism $N \to W$. In this case we know that $|P_J : P_J \cap P_J^n|$ is a power of p. We get a contradiction again. □

Step 9. *There is no counterexample to the main theorem where G is of type A_n.*

Proof. Let G be of type A_n. Then $G = PSL(n+1,q)$. Let $\overline{G} = SL(n+1,q)$, then $G = \overline{G}/Z(\overline{G})$ and there is a natural homomorphism $\rho : \overline{G} \to G$. If $H \leqslant G$, we denote the preimage of H in $\overline{G}$ by $\overline{H}$. Since $J = I - \{s, s+2\}$, $\overline{G}_x$ consists of all matrices in $\overline{G}$ of the form

$$\begin{pmatrix} M_1 & * & * \\ O & M_2 & * \\ O & O & M_3 \end{pmatrix}$$

where M_1, M_2, and M_3 are square matrices of orders s, 2, and t respectively with $s+t+2 = n+1$. We may assume that $s \neq t$, for otherwise the element of maximal length w_0 in W would send Φ_J to itself, and so $|P_J : P_J \cap P_J{}^{n_0}|$ would be a power of p, where n_0 is in the preimage of w_0 in N. This is impossible. We also assume that $s > t$.

Now consider $G_L = P_K$. By Step 5, we know that $J_1 = I - \{s, s+1, s+2\} \subseteq K$. Since for any set K_1 which properly contains J_1, the order of the group P_{K_1} is equal to or greater than the order of G_x, we have $K = J_1$. By calculating the orders of G_x and $G_L = P_{J_1}$, we know that

$$v = \frac{\prod_{i=1}^{n+1}(q^i-1)}{\prod_{i=1}^{s}(q^i-1)\prod_{i=1}^{2}(q^i-1)\prod_{i=1}^{t}(q^i-1)} = \frac{\prod_{i=t+1}^{n+1}(q^i-1)}{\prod_{i=1}^{s}(q^i-1)\prod_{i=1}^{2}(q^i-1)}$$

and

$$b = \frac{\prod_{i=1}^{n+1}(q^i-1)}{(q-1)^2\prod_{i=1}^{s}(q^i-1)\prod_{i=1}^{t}(q^i-1)}.$$

Therefore we have $b = v(q+1)$. Hence we know that $b_2 = q+1$. Recall that by definition, $b_2 = (b, v-1)$. By Lemma 2, $v - 1 = b_2k(k-1)$. So $v < b_2k^2 = (q+1)k^2$.

Now we estimate the value of k by using the method in [7]. First we consider the case when q is odd. Let $\bar{g} \in \overline{G}$ be the matrix

$$\bar{g} = \begin{pmatrix} -I_2 & O \\ O & I_{n-1} \end{pmatrix},$$

and let $\bar{g}_0 \in \overline{G}$ be the matrix

$$\bar{g}_0 = \begin{pmatrix} M_1 & & \\ & I_2 & \\ & & M_3 \end{pmatrix},$$

where I_2 is a 2×2 unit matrix, M_1 is a square matrix of order s, and M_3 a square matrix of order t, and both M_1 and M_3 have the form

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then $\bar{g}$ and $\bar{g}_0$ are conjugate to each other in $\overline{G}$, and $\bar{g}_0$ is in $\overline{G}_x$. Let g and g_0 be the images of $\bar{g}$ and $\bar{g}_0$, respectively. Then it can be easily seen that the number of conjugates of g in G is

$$w = \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}q^{2n-2}$$

and the number of conjugates of g_0 in G_x is

$$u_0 = \frac{(\prod_{i=1}^{s}(q^i-1))(\prod_{i=1}^{2}(q^i-1))(\prod_{i=1}^{t}(q^i-1))}{(q-1)^2(\prod_{i=1}^{s-1}(q^i-1))(\prod_{i=1}^{2}(q^i-1))(\prod_{i=1}^{t-1}(q^i-1))}q^{2n-2} = \frac{(q^s-1)(q^t-1)}{(q-1)(q-1)}q^{2n-2}.$$

Let u be the number of elements of G_x which are conjugate to g in G. Then $u \geqslant u_0$. By Corollary 2.4 in [6], we have

$$k < \frac{2w}{u} \leqslant \frac{2w}{u_0} = \frac{2(q^{n+1}-1)(q^n-1)(q-1)}{(q^s-1)(q^t-1)(q^2-1)}.$$

Since we know that $v < (q+1)k^2$, we have

$$v < 4\frac{(q^{n+1}-1)^2(q^n-1)^2}{(q^s-1)^2(q^t-1)^2(q+1)}.$$

Hence we get

$$\frac{\prod_{i=t+1}^{n+1}(q^i-1)}{\prod_{i=1}^{s}(q^i-1)\prod_{i=1}^{2}(q^i-1)} < 4\frac{(q^{n+1}-1)^2(q^n-1)^2}{(q^s-1)^2(q^t-1)^2(q+1)}.$$

That is,

$$\frac{\prod_{i=t}^{n-1}(q^i-1)}{(q-1)^2(\prod_{i=1}^{s-1}(q^i-1))} < 4\frac{(q^{n+1}-1)(q^n-1)}{(q^s-1)(q^t-1)}.$$

But for any two integers x and y with $x > y$, we have

$$q^{x-y} < \frac{q^x-1}{q^y-1} < 2q^{x-y}.$$

With this, we can simplify the inequality to get

$$q^{st+s+t-2} < 16q^{s+t+3} \leqslant q^{s+t+7},$$

that is,

$$st < 9.$$

From this inequality we know that there are just nine possibilities for the values of s and t. In fact, since $s > t$, we know that $t = 1$ or 2. Also, if $t = 1$, then $2 \leqslant s \leqslant 8$, and if $t = 2$, then $s = 3$ or 4. Now we show that for the case where $s = 8$ and $t = 1$ no counterexample exists, and we leave the other cases to the readers.

Since $s = 8$ and $t = 1$, we have $G = PSL(11, q)$. Then

$$\begin{aligned} v &= \frac{(q^{11}-1)(q^{10}-1)(q^9-1)}{(q^2-1)(q-1)^2} = \left(q^{10}+q^9+\cdots+1\right)\left(q^8+q^7+\cdots+1\right)\left(q^8+q^6+\cdots+1\right) \\ &\equiv 5 \pmod{q+1}. \end{aligned}$$

Since $b_2 = (q+1)|(v-1)$, we have $(q+1)|4$. Hence the only possible value for q is 3. Replacing q by 3, we can determine the values of v and b_2. But then there is no integer k satisfying $v - 1 = b_2k(k-1)$.

Now let q be even. Let $\bar{g} = \overline{g_0}$ be the matrix

$$\begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

It is easy to find out that in this case,

$$w = \frac{(q^{n+1}-1)(q^n-1)}{q-1}, \qquad u_0 = \frac{(q^s-1)(q^t-1)}{q-1}.$$

By using the same method as above we get the inequality

$$st < 11$$

and then show that no counterexample exist if q is even. We omit the details. $\square$

This completes the discussion on the cases of A_n and of D_n. In the following we assume that G is of type E_6.

Step 10. *If G_x is a maximal parabolic subgroup, then $(\mathcal{S}, G)$ is not a counterexample.*

Proof. Now let Φ be a root system of type E_6. Then we may assume that $\{e_1, \dots, e_8\}$ is an orthonormal basis of an Euclidean space $\mathcal{V}$ of dimension 8 and then $\{\alpha_1, \dots, \alpha_6\}$ constitutes a fundamental root system Φ of type E_6 (see [2]) where

$$\alpha_1 = e_3 - e_4, \qquad \alpha_2 = e_6 - e_7, \qquad \alpha_3 = e_4 - e_5,$$
$$\alpha_4 = e_5 - e_6, \qquad \alpha_5 = e_6 + e_7, \qquad \alpha_6 = -\frac{1}{2}\sum_{i=1}^{8} e_i.$$

Write the reflection r_{α_i} as r_i for $1 \leqslant i \leqslant 6$. Then $\{r_i \mid 1 \leqslant i \leqslant 6\}$ generates the Weyl group W of Φ. It is well known that $|W| = 51840$ and $W/Z_2 \cong PSU(4,3)$.

If G_x is a maximal parabolic subgroup, then $J = I - \{i\}$ for some i. If $i \in \{2,4\}$, then the longest element w_0 will send Φ_J to itself, which is impossible by Step 3. Since a graph isometry τ of Φ which interchanges α_1 and α_6 and also α_3 and α_5 induces an automorphism of G, we need only consider the cases where $J = I - \{1\}$ or $J = I - \{3\}$.

First we consider the case where $J = I - \{1\}$. Then Φ_J is a root system of type D_5 and so W_J is a semi-direct product of Z_2^4 by S_5. The intersection $\Phi_J \cap \Phi_J^{r_1}$ is a root system of type A_4. Hence $W_J \cap W_J^{r_1} \cong S_5$ and $|\, W_J : W_J \cap W_J^{r_1} \,| = 16$. Since the longest element w_0 of W sends every positive root to a negative root, $\Phi_J \cap {\Phi_J}^{w_0}$ is a root system of type D_4. From this we know that $W_J \cap W_J^{w_0}$ is a semi-direct product of Z_2^3 by S_4 and so $|W_J : W_J \cap {W_J}^{w_0}| = 10$. We get

$$W = W_J \cup W_J r_1 W_J \cup W_J w_0 W_J.$$

When we regard W as a permutation group on the set of cosets of W_J, the rank of W is 3. Since there is a bijection from the set of all double cosets $W_J w W_J$ of W to the set of all double cosets $P_J g P_J$ of G, in the action of G on the set of cosets of $\mathcal{P}_J = G_x$, G is of rank 3 and all orbits of the point stabilizer G_x are self-paired. Hence G acts on $\mathcal{S}$ flag transitively by Corollary 6, contradicting (C).

Now let $J = I - \{3\}$. Then by Step 5, we know that $K \neq I, J$ and $\{2,5,6\} \subseteq K$. If $K = I - \{1\}$, then since $|P_K| > |P_J|$, $G_L \neq P_K$. Thus we need only consider the other five subsets of I containing the subset $\{2,5,6\}$. We show that $G_L \neq P_{K_0}$ if K_0 is any one of these subsets. In fact, since G acts line transitively on $\mathcal{S}$ with a point stabilizer G_x and a line stabilizer G_L, then we have $v = |G : G_x|$ and $b = |G : G_L|$. So $b_1 = (b, v) = (|G : G_L|, |G : G_x|) = |G|/[|G_L|, |G_x|]$, where $[|G_L|, |G_x|]$ is the least common multiple of $|G_L|$ and $|G_x|$. So

$$b_2 = b/b_1 = |G : G_L| / \big(|G : G_L|, |G : G_x|\big) = \big[|G_L|, |G_x|\big]/|G_L|.$$

By the definition of b_2, we have $(b_2, v) = 1$. On the other hand, since we have supposed that $G_x = P_J$ and $J = I - \{3\}$, we have

$$v = |G : G_x| = (q+1)\big(q^2+1\big)\big(q^2-q+1\big)^2\big(q^2+q+1\big)^2\big(q^4+1\big)\big(q^4-q^2+1\big)\big(q^6+q^3+1\big).$$

Also, for every subset K_0 we can deduce the structure of the Levi subgroup L_{K_0}. Then we can easily write down the order of P_{K_0}. If for some K_0, $[|P_{K_0}|, |P_J|]/|P_{K_0}|$ is not prime to the v above, then we know that $G_L \neq P_{K_0}$.

Now we list the structures of L_{K_0} and the values of $[|P_{K_0}|, |P_J|]/|P_{K_0}|$ for all these subsets K_0 in Table 1.

Table 1

K_0	Structure of L_{K_0}	$[\lvert P_{K_0}\rvert, \lvert P_J\rvert]/\lvert P_{K_0}\rvert$
$\{2,5,6\}$	$(SL(2,q) \times SL(3,q)) : Z_{q-1}^3$	$(q+1)(q^2+1)(q^4+q^3+q^2+q+1)$
$\{1,2,5,6\}$ or $\{3,2,5,6\}$	$(SL(2,q)^2 \times SL(3,q)) : Z_{q-1}^2$	$(q^2+1)(q^4+q^3+q^2+q+1)$
$\{2,4,5,6\}$	$SL(5,q) : Z_{q-1}^2$	$(q+1)$
$\{1,2,3,5,6\}$	$SL(2,q) \times SL(3,q))^2 : Z_{q-1}$	$(q^2+1)(q^4+q^3+q^2+q+1)$

Since $q+1$ or q^2+1 is a common divisor of $[|P_{K_0}|, |P_J|]/|P_{K_0}|$ and v, G_L cannot be equal to P_{K_0} for any subset K_0 in the list. □

Step 11. *The theorem is true in all cases.*

Proof. Now we suppose that $G_x = P_J$ is not a maximal subgroup, then $|I-J| > 1$. By Step 5, the set J is not a subset of K and there is some j_0 such that $j_o \in J$ and α_{j_0} is not orthogonal to α_i for every $i \in I-J$.

In the Dynkin Diagram of type E_6, the valence of every node is at most 3. Thus we know that $|I-J| \leqslant 3$, for otherwise every root α_j with $j \in J$ will be orthogonal to at least one α_i with $i \in I-J$, but this is impossible by Step 5. Since we suppose $|I-J| > 1$, we know that $|I-J| = 2$ or 3, that is $|J| = 3$ or 4.

Now suppose that $|J| = 3$. Then $4 \in J$ since it is the only node of valence 3 and so $\{2,3,5\} \subseteq I-J$. Thus in this case $J = \{1,4,6\}$. Since the root system Φ_J is fixed under the action of w_0 if $J = \{1,4,6\}$, this case does not occur by the proof of Step 3.

Now suppose that $|J| = 4$. Since α_{j_0} is not orthogonal to any root α_i with $i \in I-J$, and $|I-J| = 2$, j_0 cannot be one of $1,2$ or 6. If $j_0 = 3$, then $1,4 \in I-J$ and so $J = \{2,3,5,6\}$. Similarly, if $j_0 = 5$, then $J = \{1,3.2,5\}$. Since a symmetry of the Dynkin diagram interchanges the subsets of nodes $\{2,3,5,6\}$ and $\{1,3.2,5\}$, we need only consider one of these cases. Suppose that $j_0 = 4$, then J is one of the following three subsets: $\{1,2,4,6\}$, $\{1,3,4,6\}$, or $\{1,4,5,6\}$. If $J = \{1,2,4,6\}$, then ${\Phi_J}^{w_0} = \Phi_J$ and we need not consider this case. Also the symmetry of the Dynkin diagram interchanges the other two subsets, thus it is sufficient to consider the case where $J = \{1,3,4,6\}$. In the following we consider two cases: (i) $J = \{2,3,5,6\}$ and (ii) $J = \{1,3,4,6\}$.

Case (i) $J = \{2,3,5,6\}$.

Take $J_1 = \{1,2,3,5,6\}$. Then $J \subset J_1$. Let $H = P_{J_1}$, a parabolic subgroup containing B. Then $G_x = P_J < H$. We can easily get that $|H : G_x| = q^2+q+1 = q(q+1)+1$. By Corollary 7, we have $b_2 | q(q+1)$. But by Gill's result we know that p is not a significant prime. Hence $b_2 | (q+1)$. But since $G_x = P_J$ is a semi-direct product of a group U_J of order a power of p by the Levi complement L_J, and the latter is a semi-direct product of $SL(2,q) \times SL(2,q) \times SL(3,q)$ by Z_{q-1}^2. We know then $v = |G : P_J|$ is a multiple of $q+1$. Thus we have that $(b_2, v) \neq 1$, a contradiction.

Case (ii) $J = \{1,3,4,6\}$.

Take $J_1 = \{1,2,3,4,6\}$ and let $H = P_{J_1}$, a parabolic subgroup containing B. Then we again have $G_x < H < G$. It is easily seen that $|H : G_x| = q(q+1)(q^2+1)+1$. Thus we have $b_2 | (q+1)(q^2+1)$ as above. But $(q+1)(q^2+1)$ is a divisor of $|G : P_J|$, we get a contradiction again.

This completes the proof of the main theorem. □

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